

STRICHARTZ ESTIMATES FOR THE WAVE EQUATION ON FLAT CONES

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ABSTRACT. We consider the solution operator for the wave equation on the flat Euclidean cone over the circle of radius $\rho > 0$, the manifold $\mathbb{R}_+ \times (\mathbb{R}/2\pi\rho\mathbb{Z})$ equipped with the metric $g(r, \theta) = dr^2 + r^2 d\theta^2$. Using explicit representations of the solution operator in regions related to flat wave propagation and diffraction by the cone point, we prove dispersive estimates and hence scale invariant Strichartz estimates for the wave equation on flat cones. We then show that this yields corresponding inequalities on wedge domains, polygons, and Euclidean surfaces with conic singularities. This in turn yields well-posedness results for the nonlinear wave equation on such manifolds. Morawetz estimates on the cone are also treated.

1. INTRODUCTION

Let $C(\mathbb{S}_\rho^1)$ denote the flat cone over the circle of radius $\rho > 0$, defined as the product manifold $C(\mathbb{S}_\rho^1) = \mathbb{R}_+ \times (\mathbb{R}/2\pi\rho\mathbb{Z})$ equipped with the (incomplete) metric $g(r, \theta) = dr^2 + r^2 d\theta^2$. In this work, we consider solutions $u : \mathbb{R} \times C(\mathbb{S}_\rho^1) \rightarrow \mathbb{C}$ to the initial value problem for the wave equation on $C(\mathbb{S}_\rho^1)$,

$$(1.1) \quad \begin{cases} (D_t^2 - \Delta_g) u(t, r, \theta) = 0 \\ u(0, r, \theta) = f(r, \theta) \\ \partial_t u(0, r, \theta) = g(r, \theta). \end{cases}$$

Here, we use Δ_g to denote the Friedrichs extension of the Laplace-Beltrami operator acting on $C_c^\infty(C(\mathbb{S}_\rho^1))$, and we write $D_t = \frac{1}{i} \partial_t$ for the Fourier-normalized time derivative.

Solutions to the wave equation on cones and related spaces have been extensively studied over the years, beginning with the seminal work of Sommerfeld [24]. In the setting of metric cones, Cheeger and Taylor [8, 9] established the propagation of singularities and provided explicit formulae for solutions in terms of the functional calculus of the Laplace-Beltrami operator on the cross-section. This was then expanded upon by Melrose and Wunsch [20], who proved a propagation of singularities theorem for solutions to the wave equation in the more general setting of conic manifolds.

Given these regularity results, it is now of interest to try to understand the related decay (or dispersive) properties of solutions. Such properties are often important in studying related nonlinear wave equations as the dispersive effect of the linear evolution can limit nonlinear interactions. Mathematically speaking, this effect gives rise to a family of space-time integrability inequalities known as *Strichartz estimates*. These estimates take the form

$$(1.2) \quad \|u\|_{L^p(\mathbb{R}; L^q(C(\mathbb{S}_\rho^1)))} \lesssim \|f\|_{\dot{H}^\gamma(C(\mathbb{S}_\rho^1))} + \|g\|_{\dot{H}^{\gamma-1}(C(\mathbb{S}_\rho^1))},$$

where the triple (p, q, γ) satisfies the scale invariant condition

$$(1.3) \quad \frac{1}{p} + \frac{2}{q} = 1 - \gamma$$

and the admissibility requirement

$$(1.4) \quad \frac{1}{p} + \frac{1}{2q} \leq \frac{1}{4}.$$

The indices are always assumed to satisfy $\gamma \geq 0$ and $2 \leq p, q \leq \infty$. Additionally, the triple $(4, \infty, \frac{3}{4})$ is excluded since Strichartz estimates are known to fail in this case.

Strichartz inequalities are well-established for constant coefficient wave equations posed on \mathbb{R}^n (see Strichartz [26], Ginibre-Velo [14], Lindblad-Sogge [19], Keel-Tao [18], and references contained therein). However, only partial progress has been made in establishing these estimates for solutions on manifolds, domains, or singular spaces such as cones. In the last case, the conic singularity affects the flow of energy and complicates many of the known techniques for establishing these inequalities. Nonetheless, in [13] the second author developed a representation of the fundamental solution to the Schrödinger equation on $C(\mathbb{S}_\rho^1)$ and used it to prove the analogous Strichartz estimates for that equation. The present authors together with Sebastian Herr then used this theorem to obtain estimates for the Schrödinger equation on polygonal domains in [2]. The main idea in [13] was to use the functional calculus on cones developed by Cheeger [6, 7] to calculate an explicit representation of the Schwartz kernel of $\exp(it\Delta_g)$. In particular, it was shown that the kernel is uniformly bounded by $|t|^{-1}$, and hence there is the dispersive estimate

$$(1.5) \quad \|\exp(it\Delta_g) f\|_{L^\infty(C(\mathbb{S}_\rho^1))} \lesssim |t|^{-1} \|f\|_{L^1(C(\mathbb{S}_\rho^1))}.$$

Such a dispersive estimate is the key to establishing the full range of Strichartz inequalities for the Schrödinger equation.

Explicit representations of the fundamental solution to the wave equation on $C(\mathbb{S}_\rho^1)$ were developed by Cheeger and Taylor in [9, Section 4]. Unlike the fundamental solution to the Schrödinger equation, however, the fundamental solution to the wave equation is unbounded, and as a consequence one must rework the $L^1 \rightarrow L^\infty$ dispersive estimates. Even on \mathbb{R}^2 , there is no estimate analogous to (1.5) which is valid for any choice of initial data, regardless of whether derivatives are incorporated to ensure scale invariance. One way to circumvent this problem is to prove $L^1 \rightarrow L^\infty$ estimates for frequency localized solutions, showing that whenever the initial data (f, g) is spectrally localized to frequencies near $\mu > 0$ there is a replacement for the analogue of (1.5). Namely, Strichartz estimates may be proved if one shows the following.

Conjecture 1.1. *Suppose $\beta(\mu^{-1}\sqrt{\Delta_g}) f = f$ and $\beta(\mu^{-1}\sqrt{\Delta_g}) g = g$ for some smooth cutoff β with $\text{supp}(\beta) \subset (\frac{1}{4}, 4)$. Then*

$$(1.6) \quad \|\mathcal{U}(t)g\|_{L^\infty(C(\mathbb{S}_\rho^1))} \lesssim \mu(1 + \mu|t|)^{-1/2} \|g\|_{L^1(C(\mathbb{S}_\rho^1))}$$

$$(1.7) \quad \|\dot{\mathcal{U}}(t)f\|_{L^\infty(C(\mathbb{S}_\rho^1))} \lesssim \mu(1 + \mu|t|)^{-1/2} \left(\mu \|f\|_{L^1(C(\mathbb{S}_\rho^1))} + \|\nabla_g f\|_{L^1(C(\mathbb{S}_\rho^1))} \right),$$

where we use the abbreviations

$$(1.8) \quad \mathcal{U}(t) \stackrel{\text{def}}{=} \frac{\sin\left(t\sqrt{\Delta_g}\right)}{\sqrt{\Delta_g}} \quad \text{and} \quad \dot{\mathcal{U}}(t) \stackrel{\text{def}}{=} \cos\left(t\sqrt{\Delta_g}\right).$$

Our main theorem in this work is that the former estimate (1.6) holds and, by making use of the Hilbert transform, that this estimate is sufficient to yield the full range of Strichartz estimates.

Theorem 1.2. *Suppose $(f, g) \in \dot{H}^\gamma(C(\mathbb{S}_\rho^1)) \times \dot{H}^{\gamma-1}(C(\mathbb{S}_\rho^1))$. Then,*

$$(1.9) \quad \|\mathcal{U}(t)g\|_{L^p(\mathbb{R}; L^q(C(\mathbb{S}_\rho^1)))} \lesssim \|g\|_{\dot{H}^{\gamma-1}(C(\mathbb{S}_\rho^1))},$$

$$(1.10) \quad \|\dot{\mathcal{U}}(t)f\|_{L^p(\mathbb{R}; L^q(C(\mathbb{S}_\rho^1)))} \lesssim \|f\|_{\dot{H}^\gamma(C(\mathbb{S}_\rho^1))},$$

for any triple (p, q, γ) satisfying (1.3), (1.4), and $2 \leq p, q \leq \infty$ provided $(p, q, \gamma) \neq (4, \infty, \frac{3}{4})$. Hence, solutions u to the wave equation IVP (1.1) satisfy the Strichartz estimates (1.2).

We define the homogeneous Sobolev spaces appearing in this theorem via the spectral resolution of the Laplacian; the details are discussed in Section 2.

Using duality and an application of the Christ-Kiselev lemma [11], we will also establish inhomogeneous Strichartz estimates.

Corollary 1.3. *Suppose $2 \leq p, q < \infty$ and $2 \leq \tilde{p}, \tilde{q} < \infty$ satisfy (1.4) and that*

$$(1.11) \quad \frac{1}{p} + \frac{2}{q} = \frac{1}{\tilde{p}'} + \frac{2}{\tilde{q}'} - 2 = 1 - \gamma$$

with $(\cdot)'$ denoting the Hölder-dual exponent, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. If the inhomogeneity F is in $L^{\tilde{p}'}(\mathbb{R}; L^{\tilde{q}'}(C(\mathbb{S}_\rho^1)))$ and

$$w(t, r, \theta) = \int_{-\infty}^t (\mathcal{U}(t-s)F(s, \cdot))(r, \theta) ds,$$

then

$$(1.12) \quad \|w\|_{L^p(\mathbb{R}; L^q(C(\mathbb{S}_\rho^1)))} + \|(w, \partial_t w)\|_{L^\infty(\mathbb{R}; \dot{H}^\gamma(C(\mathbb{S}_\rho^1)) \times \dot{H}^{\gamma-1}(C(\mathbb{S}_\rho^1)))} \lesssim \|F\|_{L^{\tilde{p}'}(\mathbb{R}; L^{\tilde{q}'}(C(\mathbb{S}_\rho^1)))}.$$

Note that by localizing the driving force to $[0, \infty)$, we obtain estimates for w satisfying $(D_t^2 - \Delta_g)w = F$ with vanishing initial data.

As a byproduct of the proof of Theorem 1.2, we also show local estimates on solutions which instead involve inhomogeneous Sobolev spaces on the right-hand side. These estimates will play a role in Section 4, where local estimates on planar domains are developed.

Corollary 1.4. *Let u be a solution to the inhomogeneous problem,*

$$(1.13) \quad \begin{cases} (D_t^2 - \Delta_g)u(t, r, \theta) = F(t, r, \theta) \\ u(0, r, \theta) = f(r, \theta) \\ \partial_t u(0, r, \theta) = g(r, \theta). \end{cases}$$

Suppose that $2 \leq p, q < \infty$ and $2 \leq \tilde{p}, \tilde{q} < \infty$ satisfy (1.4) and (1.11). Then for some implicit constant depending on T ,

$$(1.14) \quad \|u\|_{L^p([-T, T]; L^q(C(\mathbb{S}_\rho^1)))} + \|(u, \partial_t u)\|_{L^\infty([-T, T]; H^\gamma(C(\mathbb{S}_\rho^1)) \times H^{\gamma-1}(C(\mathbb{S}_\rho^1)))} \\ \lesssim \|(f, g)\|_{H^\gamma(C(\mathbb{S}_\rho^1)) \times H^{\gamma-1}(C(\mathbb{S}_\rho^1))} + \|F\|_{L^{\tilde{p}'}([-T, T]; L^{\tilde{q}'}(C(\mathbb{S}_\rho^1)))},$$

whenever the right hand side is finite.

The expressions for the Schwartz kernels of $\mathcal{U}(t)$ derived by Cheeger and Taylor vary depending on into which of three regions of the spacetime $\mathbb{R} \times C(\mathbb{S}_\rho^1) \times C(\mathbb{S}_\rho^1)$ their arguments fall; these regions are

$$(1.15) \quad \begin{aligned} \text{Region I} &\stackrel{\text{def}}{=} \{(t, r_1, \theta_1, r_2, \theta_2) : 0 < t < d_g((r_1, \theta_1), (r_2, \theta_2))\}, \\ \text{Region II} &\stackrel{\text{def}}{=} \{(t, r_1, \theta_1, r_2, \theta_2) : d_g((r_1, \theta_1), (r_2, \theta_2)) < t < r_1 + r_2\}, \text{ and} \\ \text{Region III} &\stackrel{\text{def}}{=} \{(t, r_1, \theta_1, r_2, \theta_2) : t > r_1 + r_2\}. \end{aligned}$$

Here, the Riemannian distance function d_g is (see [8, (3.41)])

$$(1.16) \quad d_g((r_1, \theta_1), (r_2, \theta_2)) = \begin{cases} (r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2))^{1/2}, & |\theta_1 - \theta_2| \leq \pi \\ r_1 + r_2, & |\theta_1 - \theta_2| \geq \pi, \end{cases}$$

provided the angular coordinates θ_1 and θ_2 are chosen so that $|\theta_1 - \theta_2|$ gives the distance between the two points on the circle. See Figure 1 for a heuristic sketch of these regions in \mathbb{R}^2 at a point in time when Region III is nonempty. Region I is the part of spacetime in which the propagator is identically zero owing to finite speed of the propagation of supports. Region II is the regime in which waves propagate as they would on a smooth manifold, i.e. the region in which there has been no interaction between the main front and the cone tip. Finally, Region III is the region in which there has been an interaction between the main front and the cone point. The singularities, i.e. wavefront set, of the propagators in the transition between Regions I and II are entirely “geometric”, to use the terminology of Melrose and Wunsch [20], which is to say that they propagate via geodesic flow. Those in the transition between Regions II and III can be either geometric or “diffractive”. Heuristically speaking, the geometric singularities in this transition are the limits of the geometric singularities in the transition between Regions I and II, and the diffractive singularities are those emerging radially from the cone point after a singularity has entered.

We now discuss the formulae for the Schwartz kernel of $\mathcal{U}(t)$ in the different regions of spacetime. As we remarked, in Region I

$$(1.17) \quad K_{\mathcal{U}(t)}^I(r_1, \theta_1; r_2, \theta_2) \equiv 0.$$

In Region II, it is given by

$$(1.18) \quad \begin{aligned} K_{\mathcal{U}(t)}^{II}(r_1, \theta_1; r_2, \theta_2) \\ = \frac{1}{2\pi} \sum_j [t^2 - r_1^2 - r_2^2 + 2r_1r_2 \cos(\theta_1 - \theta_2 - j \cdot 2\pi\rho)]^{-\frac{1}{2}}, \end{aligned}$$

where the index j ranges over integers such that

$$(1.19) \quad 0 < |\theta_1 - \theta_2 - j \cdot 2\pi\rho| < \cos^{-1} \left(\frac{r_1^2 + r_2^2 - t^2}{2r_1r_2} \right).$$

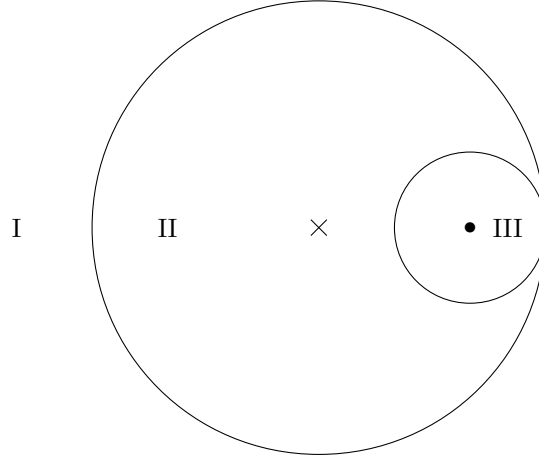


FIGURE 1. The Regions of behavior; “×” denotes the initial pole of the source and “•” the cone point

In Region III, it is given by

$$\begin{aligned}
 (1.20) \quad & K_{\mathcal{U}(t)}^{\text{III}}(r_1, \theta_1; r_2, \theta_2) \\
 &= \frac{1}{2\pi} \sum_j [t^2 - r_1^2 - r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2 - j \cdot 2\pi\rho)]^{-\frac{1}{2}} \\
 &\quad - \frac{1}{4\pi^2\rho} \int_0^{\cosh^{-1}\left(\frac{t^2 - r_1^2 - r_2^2}{2r_1 r_2}\right)} [t^2 - r_1^2 - r_2^2 - 2r_1 r_2 \cosh(s)]^{-\frac{1}{2}} \\
 &\quad \times \left\{ \frac{\sin\left[\frac{\pi + \theta_1 - \theta_2}{\rho}\right]}{\cosh\left[\frac{s}{\rho}\right] - \cos\left[\frac{\theta_1 - \theta_2 + \pi}{\rho}\right]} + \frac{\sin\left[\frac{\pi - (\theta_1 - \theta_2)}{\rho}\right]}{\cosh\left[\frac{s}{\rho}\right] - \cos\left[\frac{\theta_1 - \theta_2 - \pi}{\rho}\right]} \right\} ds,
 \end{aligned}$$

where j now ranges over

$$(1.21) \quad 0 < |\theta_1 - \theta_2 - j \cdot 2\pi\rho| < \pi.$$

The remainder of the paper is organized as follows. In Section 2, we review Cheeger’s functional calculus for metric cones and define the corresponding homogeneous Sobolev spaces. Section 3 addresses the Strichartz estimates in Theorem 1.2. In Section 4, we conclude by exploring applications and extensions of these estimates. Specifically, we discuss Morawetz estimates on the Euclidean cone as well as Strichartz estimates on wedges, polygons, and Euclidean surfaces with conic singularities. We then apply these estimates to the well-posedness of nonlinear wave equations with initial data of minimal regularity.

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2. SPECTRAL THEORY AND FUNCTION SPACES

We begin by briefly recalling Cheeger's functional calculus for metric cones $C(Y)$. Using this, we define the homogeneous Sobolev spaces $\dot{H}^s(C(Y))$ appearing in the Strichartz estimates (1.2). We refer to Cheeger's articles with Taylor [8, 9] or the second book of Taylor's series [27] for an in-depth discussion of the functional calculus as well as other applications.

2.1. Cheeger's functional calculus. Let Y^n be a compact, boundaryless Riemannian manifold with metric h , and let $C(Y) \stackrel{\text{def}}{=} \mathbb{R}_+ \times Y$ be the half-cylinder over Y . To make $C(Y)$ into a metric cone, we equip it with the incomplete Riemannian metric

$$(2.1) \quad g(r, y) = dr^2 + r^2 h(y).$$

The nonnegative Laplacian on $C(Y)$ is thus

$$(2.2) \quad \Delta_g = -\partial_r^2 - \frac{n}{r} \partial_r + \frac{1}{r^2} \Delta_h,$$

where Δ_h is the nonnegative Laplacian on the cross-section Y . Writing $\{\mu_j\}_{j=0}^\infty$ for the eigenvalues of Δ_h with multiplicity and $\{\varphi_j : Y \rightarrow \mathbb{C}\}_{j=0}^\infty$ for the corresponding orthonormal basis of eigenfunctions, we define the rescaled eigenvalues ν_j by

$$(2.3) \quad \nu_j \stackrel{\text{def}}{=} \left(\mu_j + \frac{(n-1)^2}{4} \right)^{\frac{1}{2}}.$$

Note that $\mu_0 = 0$ and $\nu_0 = \frac{n-1}{2}$ in our convention.

Henceforth, we take Δ_g to be the Friedrichs extension of the above Laplace-Beltrami operator on functions. As is well-known, suitable functions $G : \mathbb{R} \rightarrow \mathbb{C}$ give rise to operators $G(\Delta_g)$ via the spectral theorem. By taking advantage of the product structure of the metric $g(r, y)$ and separation of variables, Cheeger showed that the Schwartz kernel of $G(\Delta_g)$, which we will write as $K_{G(\Delta_g)}$, has the form

$$(2.4) \quad K_{G(\Delta_g)}(r_1, y_1; r_2, y_2) = (r_1 r_2)^{-\frac{n-1}{2}} \sum_{j=0}^{\infty} \tilde{K}_{G(\Delta_g)}(r_1, r_2, \nu_j) \varphi_j(y_1) \overline{\varphi_j(y_2)},$$

where the radial coefficient $\tilde{K}_{G(\Delta_g)}(r_1, r_2, \nu_j)$ is given by

$$(2.5) \quad \tilde{K}_{G(\Delta_g)}(r_1, r_2, \nu) \stackrel{\text{def}}{=} \int_0^\infty G(\lambda^2) J_\nu(\lambda r_1) J_\nu(\lambda r_2) \lambda d\lambda.$$

Here, $J_\nu(z)$ is the Bessel function of order ν ,

$$J_\nu(z) \stackrel{\text{def}}{=} \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(\nu + j + 1)} \left(\frac{z}{2} \right)^{\nu+2j}.$$

We can view the formulae (2.4) and (2.5) as consequences of a “factoring” of the spectral measure on $C(Y)$ into tangential components and radial components. Indeed, the product $J_{\nu_j}(\lambda r)\varphi_j(y)$ is a solution to the Helmholtz equation

$$(\Delta_g - \lambda^2) (J_{\nu_j}(\lambda r)\varphi_j(y)) = 0.$$

This naturally leads one to consider the Hankel transform of order ν_j ,

$$(2.6) \quad \mathcal{H}_{\nu_j}[b(r)](\lambda) \stackrel{\text{def}}{=} \int_0^\infty b(r) J_{\nu_j}(\lambda r) r dr,$$

a unitary map $L^2(\mathbb{R}_+, r dr) \longrightarrow L^2(\mathbb{R}_+, \lambda d\lambda)$ satisfying $\mathcal{H}_{\nu_j} \circ \mathcal{H}_{\nu_j} = \text{Id}$ (see [27, Ch. 8, Proposition 8.1]). This gives rise to a unitary isomorphism

$$(2.7) \quad \mathcal{H} : L^2(C(Y)) \xrightarrow{\cong} L^2(\mathbb{R}_+, \lambda d\lambda; \ell^2(\mathbb{Z}_{\geq 0}))$$

which acts on functions $a(r, y)$ in $L^2(C(Y))$ by

$$(2.8) \quad \mathcal{H}[a](\lambda) \stackrel{\text{def}}{=} ((\mathcal{H}_{\nu_j} \circ \Pi_j)[a(\cdot, \cdot)](\lambda))_{j \in \mathbb{Z}},$$

where Π_j is the projection onto the j -th eigenfunction of the tangential Laplacian Δ_h ,

$$(2.9) \quad \Pi_j[a(\cdot, \cdot)](r) \stackrel{\text{def}}{=} \int_Y a(r, y) \overline{\varphi_j(y)} dh.$$

For further details, we refer the reader to Cheeger and Taylor [8].

2.2. The Sobolev spaces. We now define the function spaces needed for our analysis.

Definition 2.1. For $s \geq 0$, we define the *homogeneous Sobolev spaces* $\dot{H}^s(C(Y))$ to be the completion of $\mathcal{C}_c^\infty(C(Y))$ in the topology induced by the inner product

$$(2.10) \quad \langle u, v \rangle_{\dot{H}^s} \stackrel{\text{def}}{=} \left\langle \Delta_g^{\frac{s}{2}} u, \Delta_g^{\frac{s}{2}} v \right\rangle_{L^2}.$$

We define the homogeneous Sobolev spaces of negative order by duality, i.e. for $s > 0$,

$$(2.11) \quad \dot{H}^{-s}(C(Y)) \stackrel{\text{def}}{=} \left(\dot{H}^s(C(Y)) \right)',$$

equipped with the dual norm.

When $s > 0$, functions in $\text{Dom}(\Delta_g^{-\frac{s}{2}})$ define elements in $\dot{H}^{-s}(C(Y))$ via the usual L^2 -pairing. Indeed, if $\psi \in \text{Dom}(\Delta_g^{-\frac{s}{2}})$ and $f \in \dot{H}^s(C(Y)) \cap \text{Dom}(\Delta_g^{\frac{s}{2}})$, then the functional calculus shows that $\Delta_g^{-\frac{s}{2}}\psi \in \text{Dom}(\Delta_g^{\frac{s}{2}})$; hence

$$|\langle f, \psi \rangle_{L^2}| = \left| \left\langle \Delta_g^{\frac{s}{2}} f, \Delta_g^{-\frac{s}{2}} \psi \right\rangle_{L^2} \right| \leq \|f\|_{\dot{H}^s} \left\| \Delta_g^{-\frac{s}{2}} \psi \right\|_{L^2}.$$

This also shows that if $\Psi \in \dot{H}^{-s}(C(Y))$ is defined by $\Psi(f) = \langle f, \psi \rangle$, then we can take $f = \Delta_g^{-\frac{s}{2}}\psi$ to see that $\|\Psi\|_{\dot{H}^{-s}} = \left\| \Delta_g^{-\frac{s}{2}} \psi \right\|_{L^2}$. Moreover, $\text{Dom}(\Delta_g^{-\frac{s}{2}})$ is dense in $\dot{H}^{-s}(C(Y))$. Indeed, given any $\Psi \in \dot{H}^{-s}(C(Y))$, the Riesz representation theorem shows that there exists $\phi \in \dot{H}^s(C(Y))$ such that $\Psi(f) = \langle f, \phi \rangle_{\dot{H}^s}$ and $\|\Psi\|_{\dot{H}^{-s}} = \|\phi\|_{\dot{H}^s}$. Since $\mathcal{C}_c^\infty(C(Y))$ is dense in $\text{Dom}(\Delta_g^{\frac{s}{2}})$, there exists a sequence $\{\phi_n\}_{n=1}^\infty \subseteq \mathcal{C}_c^\infty(C(Y))$ such that $\|\phi_n - \phi\|_{\dot{H}^s} \longrightarrow 0$. Thus $\psi_n \stackrel{\text{def}}{=} \Delta_g^s \phi_n$ is well

defined and provides a sequence of functions in $\text{Dom}\left(\Delta_g^{-\frac{s}{2}}\right)$ which approximates Ψ in $\dot{H}^{-s}(C(Y))$.

As a result of this density property, we may take Y to be the 1-dimensional manifold \mathbb{S}_ρ^1 to see that it suffices to show Theorem 1.2 under the stronger assumption

$$(2.12) \quad (f, g) \in \left[\dot{H}^\gamma(C(\mathbb{S}_\rho^1)) \cap \text{Dom}\left(\Delta_g^{\frac{\gamma}{2}}\right) \right] \times \left[\dot{H}^{\gamma-1}(C(\mathbb{S}_\rho^1)) \cap \text{Dom}\left(\Delta_g^{\frac{\gamma-1}{2}}\right) \right].$$

We define the inhomogeneous Sobolev spaces similarly.

Definition 2.2. For all real s , we define the *inhomogeneous Sobolev spaces*, denoted $H^s(C(Y))$, to be the closure of $\mathcal{C}_c^\infty(C(Y))$ in the topology induced by the inner product

$$(2.13) \quad \langle u, v \rangle_{H^s} \stackrel{\text{def}}{=} \left\langle (\text{Id} + \Delta_g)^{\frac{s}{2}} u, (\text{Id} + \Delta_g)^{\frac{s}{2}} v \right\rangle_{L^2}.$$

3. DISPERSIVE ESTIMATES

In this section, we prove Theorem 1.2 using an explicit formula for the kernel of the sine propagator $\mathcal{U}(t)$. As discussed in the introduction, we will take a Littlewood-Paley decomposition of the solution to establish frequency-localized dispersive estimates. These will allow for a regularization of $K_{\mathcal{U}(t)}$ to unit frequency that will overcome the complications coming from unboundedness of the fundamental solution.

3.1. Reduction to dispersive estimates. Let $\{\beta_k\}_{k \in \mathbb{Z}}$ be a collection of smooth cutoffs satisfying

$$(3.1) \quad \sum_{k=-\infty}^{\infty} \beta_k(\zeta) \equiv 1, \quad \beta_k(\zeta) \stackrel{\text{def}}{=} \beta_0(2^{-k}\zeta), \quad \text{and} \quad \text{supp}(\beta_0) \subset \left(\frac{1}{\sqrt{2}}, 2\sqrt{2} \right).$$

We define the associated Littlewood-Paley frequency cutoffs $\beta_k(\sqrt{\Delta_g})$ using the functional calculus reviewed in Section 2, and we see immediately from the definition that

$$\sum_{k=-\infty}^{\infty} \beta_k(\sqrt{\Delta_g}) = \text{Id} : L^2(C(\mathbb{S}_\rho^1)) \longrightarrow L^2(C(\mathbb{S}_\rho^1)).$$

More generally, these Littlewood-Paley cutoffs satisfy a range of squarefunction estimates, which we summarize in the following proposition.

Proposition 3.1. *Let $1 < q < \infty$. For elements $a \in L^q(C(\mathbb{S}_\rho^1))$, we have*

$$(3.2) \quad \left\| \left(\sum_{k=-\infty}^{\infty} \left| \beta_k(\sqrt{\Delta_g}) a \right|^2 \right)^{\frac{1}{2}} \right\|_{L^q(C(\mathbb{S}_\rho^1))} \approx \|a\|_{L^q(C(\mathbb{S}_\rho^1))},$$

with implicit constants depending only on q .

The proof of this proposition is implicit in the arguments in Section 4 of [2], the authors' previous paper with Herr, where the squarefunction estimates are shown for functions on a Euclidean surface with conical singularities. Namely, (3.2) follows from the spectral multiplier theorem of Alexopolous [1], which is valid in any context where the heat kernel satisfies Gaussian upper bounds. A result of Grigor'yan [16] shows that such bounds are true provided they are satisfied along the diagonal. On

the cone $C(\mathbb{S}_\rho^1)$, these on-diagonal bounds follow from an explicit formula for the heat kernel. See Section 4 of [2] for a more thorough discussion.

Returning to the wave equation, suppose u is a solution to the IVP (1.1) with initial data satisfying (2.12). We define its frequency decomposition to be

$$(3.3) \quad u_k(t, \cdot) \stackrel{\text{def}}{=} \beta_k(\sqrt{\Delta_g}) u(t, \cdot) \quad \text{and} \quad (f_k, g_k) \stackrel{\text{def}}{=} (\beta_k(\sqrt{\Delta_g}) f, \beta_k(\sqrt{\Delta_g}) g).$$

Since the frequency cutoffs commute with the Laplacian, the frequency localized solutions $\{u_k\}_{k \in \mathbb{Z}}$ satisfy the collection of IVPs

$$(3.4) \quad \begin{cases} (D_t^2 - \Delta_g) u_k(t, r, \theta) = 0 \\ u_k(0, r, \theta) = f_k(r, \theta) \\ \partial_t u_k(0, r, \theta) = g_k(r, \theta). \end{cases}$$

As a consequence of the squarefunction estimates (3.2) and Minkowski's inequality we have

$$\|u\|_{L^p(\mathbb{R}; L^q(C(\mathbb{S}_\rho^1)))} \lesssim \left(\sum_{k=-\infty}^{\infty} \|u_k\|_{L^p(\mathbb{R}; L^q(C(\mathbb{S}_\rho^1)))}^2 \right)^{\frac{1}{2}}.$$

Furthermore, we note that the operator norm of

$$2^{-ks} \Delta_g^{\frac{s}{2}} \beta_k(\sqrt{\Delta_g})$$

on $L^2(C(\mathbb{S}_\rho^1))$ is uniformly bounded in k , which implies that

$$(3.5) \quad 2^{k\gamma} \|f_k\|_{L^2} + 2^{k(\gamma-1)} \|g_k\|_{L^2} \lesssim \left\| \Delta_g^{\frac{\gamma}{2}} f_k \right\|_{L^2} + \left\| \Delta_g^{\frac{\gamma-1}{2}} g_k \right\|_{L^2}.$$

Therefore, if we show the collection of frequency-localized Strichartz estimates

$$(3.6) \quad \|u_k\|_{L^p(\mathbb{R}; L^q(C(\mathbb{S}_\rho^1)))} \lesssim 2^{k\gamma} \|f_k\|_{L^2(C(\mathbb{S}_\rho^1))} + 2^{k(\gamma-1)} \|g_k\|_{L^2(C(\mathbb{S}_\rho^1))},$$

then the desired Strichartz estimates (1.2) will follow from (3.5) and the bound

$$(3.7) \quad \begin{aligned} & \sum_{k=-\infty}^{\infty} \left(\left\| \Delta_g^{\frac{\gamma}{2}} f_k \right\|_{L^2}^2 + \left\| \Delta_g^{\frac{\gamma-1}{2}} g_k \right\|_{L^2}^2 \right) \\ &= \sum_{k=-\infty}^{\infty} \left(\left\| \beta_k(\sqrt{\Delta_g}) \Delta_g^{\frac{\gamma}{2}} f \right\|_{L^2}^2 + \left\| \beta_k(\sqrt{\Delta_g}) \Delta_g^{\frac{\gamma-1}{2}} g \right\|_{L^2}^2 \right) \\ &\lesssim \|f\|_{H^\gamma}^2 + \|g\|_{H^{\gamma-1}}^2. \end{aligned}$$

Remark 3.2. We also note at this stage that the homogeneous estimates in Corollary 1.4 follow by a slight modification of these arguments. A corresponding adjustment in the proof of Corollary 1.3 will handle the inhomogeneous inequality.

For local estimates, we instead take $\tilde{u}_0 \stackrel{\text{def}}{=} \sum_{k \leq 0} u_k$, and we observe by Sobolev embedding and the fact that \tilde{u}_0 is supported at low frequency that

$$\begin{aligned} \|\tilde{u}_0\|_{L^p([-T, T]; L^q(C(\mathbb{S}_\rho^1)))} &\lesssim \sup_{-T \leq t \leq T} \|\tilde{u}_0(t)\|_{H^{\gamma+\frac{1}{2}}(C(\mathbb{S}_\rho^1))} \\ &\lesssim \|f\|_{H^\gamma(C(\mathbb{S}_\rho^1))} + \|g\|_{H^{\gamma-1}(C(\mathbb{S}_\rho^1))}. \end{aligned}$$

Now, note that we can replace Δ_g by $\text{Id} + \Delta_g$ in (3.5) when $k \geq 1$. Thus if (3.6) holds, we can apply reasoning similar to the above to prove Corollary 1.4.

We now reduce the collection of frequency-localized Strichartz estimates (3.6) to a single Strichartz estimate for initial data with frequency localized to unit scale.

Lemma 3.3. *Suppose $f, g \in L^2(C(\mathbb{S}_\rho^1))$ are functions satisfying*

$$\beta(\sqrt{\Delta_g}) f = f \quad \text{and} \quad \beta(\sqrt{\Delta_g}) g = g$$

for some smooth cutoff $\beta \in C_c^\infty(\mathbb{R})$ supported in $(\frac{1}{4}, 4)$. Suppose also that the corresponding solution to the wave equation IVP (1.1) satisfies the Strichartz estimate

$$(3.8) \quad \|u\|_{L^p(\mathbb{R}; L^q(C(\mathbb{S}_\rho^1)))} \lesssim \|f\|_{L^2(C(\mathbb{S}_\rho^1))} + \|g\|_{L^2(C(\mathbb{S}_\rho^1))}.$$

Then the estimates (3.6) hold for all integers k .

Proof. Recall that the wave equation is invariant under the scaling

$$(t, r, \theta) \mapsto (\mu^{-1}t, \mu^{-1}r, \theta).$$

It thus suffices to show that given h in $L^2(C(\mathbb{S}_\rho^1))$, the rescaled function

$$(\beta_k(\sqrt{\Delta_g}) h)(2^{-k}r, \theta)$$

is localized to unit frequency. Let $\varphi_j(\theta) \stackrel{\text{def}}{=} (2\pi\rho)^{-\frac{1}{2}} e^{ij\theta/\rho}$ be a normalized eigenfunction on \mathbb{S}_ρ^1 . Without loss of generality, we can assume that h takes the form $h(r, \theta) = h_j(r) \varphi_j(\theta)$, since any $h \in L^2(C(\mathbb{S}_\rho^1))$ can be realized as a sum of such functions. Therefore,

$$\begin{aligned} & (\beta_k(\sqrt{\Delta_g}) h)(2^{-k}r, \theta) \\ &= \varphi_j(\theta) \int_0^\infty \int_0^\infty \beta_0(2^{-k}\lambda) J_{\nu_j}(2^{-k}\lambda r) J_{\nu_j}(\lambda s) h_j(s) s ds \lambda d\lambda. \end{aligned}$$

We now make the change of variables $(\lambda, s) \mapsto (2^k\lambda, 2^{-k}s)$ in the integral on the right, producing

$$\varphi_j(\theta) \int_0^\infty \int_0^\infty \beta_0(\lambda) J_{\nu_j}(\lambda r) J_{\nu_j}(\lambda s) h_j(2^{-k}s) s ds \lambda d\lambda.$$

This is $(\beta_0(\sqrt{\Delta_g}) h(2^{-k}\cdot, \cdot))(r, \theta)$, which is manifestly of the desired form. \square

Hence, tracing back through these reductions, we see that the main theorem is a consequence of the following two estimates for initial data (f, g) that is localized to unit frequency and any smooth cutoff β supported in $(\frac{1}{4}, 4)$:

$$(3.9) \quad \left\| \beta(\sqrt{\Delta_g}) \mathcal{U}(t) g \right\|_{L^p(\mathbb{R}; L^q(C(\mathbb{S}_\rho^1)))} \lesssim \|g\|_{L^2(C(\mathbb{S}_\rho^1))}$$

$$(3.10) \quad \left\| \beta(\sqrt{\Delta_g}) \dot{\mathcal{U}}(t) f \right\|_{L^p(\mathbb{R}; L^q(C(\mathbb{S}_\rho^1)))} \lesssim \left\| \sqrt{\Delta_g} f \right\|_{L^2(C(\mathbb{S}_\rho^1))}.$$

Remark 3.4. The main theorem follows from (3.9) alone, for the Strichartz estimate (3.10) for the cosine evolution operator is a consequence of that for the sine evolution operator. To see this, let \mathcal{T} denote the operator acting on functions $v : \mathbb{R} \times C(\mathbb{S}_\rho^1) \rightarrow \mathbb{R}$ by taking the Hilbert transform in the first variable, i.e.

$$(3.11) \quad (\mathcal{T}v)(t, r, \theta) \stackrel{\text{def}}{=} \frac{1}{\pi} \cdot \text{PV} \int_{-\infty}^\infty \frac{v(s, r, \theta)}{t - s} ds$$

where PV denotes that this is a principal value integral. Writing

$$\begin{aligned} w(t, r, \theta) &\stackrel{\text{def}}{=} \left(\beta(\sqrt{\Delta_g}) \dot{\mathcal{U}}(t) f \right)(r, \theta) \\ v(t, r, \theta) &\stackrel{\text{def}}{=} \left(\beta(\sqrt{\Delta_g}) \mathcal{U}(t) \sqrt{\Delta_g} f \right)(r, \theta) \end{aligned}$$

and \mathcal{H} for the spectral resolution of Δ_g as in Section 2, we have

$$\begin{aligned} (\mathcal{H}w)(t, \lambda) &= \beta(\lambda) \cos(t\lambda) \cdot (\mathcal{H}f)(\lambda) \\ &= -\frac{1}{\pi} \cdot \text{PV} \int_{-\infty}^{\infty} \frac{\sin(t\lambda) \beta(\lambda) (\mathcal{H}f)(\lambda)}{t-s} ds \\ &= -\mathcal{T}[(\mathcal{H}v)(t, \lambda)]. \end{aligned}$$

Thus, $w = -\mathcal{T}v$ since the Hilbert transform commutes with \mathcal{H} . Our claim then follows from noting that \mathcal{T} is bounded as an operator on $L^p(\mathbb{R}; L^q(C(\mathbb{S}_\rho^1)))$, implying the estimate

$$\left\| \beta(\sqrt{\Delta_g}) \dot{\mathcal{U}}(t) f \right\|_{L^p(\mathbb{R}; L^q(C(\mathbb{S}_\rho^1)))} \lesssim \left\| \beta(\sqrt{\Delta_g}) \mathcal{U}(t) \sqrt{\Delta_g} f \right\|_{L^p(\mathbb{R}; L^q(C(\mathbb{S}_\rho^1)))}.$$

We note that the bound on \mathcal{T} is a consequence of the Calderón-Zygmund theory of vector-valued singular integrals; see, for instance, Theorem 4.6.1 in Grafakos' book [15]. With this remark in mind, we now prove that Corollary 1.3 is a consequence of (3.9).

Proof of Corollary 1.3. Let

$$\mathcal{S}_1 \stackrel{\text{def}}{=} \frac{\beta(\sqrt{\Delta_g}) e^{it\sqrt{\Delta_g}}}{\sqrt{\Delta_g}} \quad \text{and} \quad \mathcal{S}_2 \stackrel{\text{def}}{=} \beta(\sqrt{\Delta_g}) e^{it\sqrt{\Delta_g}}.$$

By Euler's formula and the remark above (changing the form of β if necessary), we have that if $\beta(\sqrt{\Delta_g}) f = f$, then

$$\|\mathcal{S}_i f\|_{L^p(\mathbb{R}; L^q(C(\mathbb{S}_\rho^1)))} \lesssim \|f\|_{L^2(C(\mathbb{S}_\rho^1))}$$

for $i = 1, 2$ and any admissible (p, q) in (1.4), including $(p, q) = (\infty, 2)$. Now take $F(s, r, \theta)$ to be a smooth function, compactly supported in time, such that $\beta(\sqrt{\Delta_g}) F(s, \cdot) = F(s, \cdot)$. By duality, if (\tilde{p}, \tilde{q}) is also admissible then we have

$$\|\mathcal{S}_2 \mathcal{S}_1^* F\|_{L^p(\mathbb{R}; L^q(C(\mathbb{S}_\rho^1)))} \lesssim \|F\|_{L^{\tilde{p}'}(\mathbb{R}; L^{\tilde{q}'}(C(\mathbb{S}_\rho^1)))}.$$

The Christ-Kiselev lemma [11] and time reversal now give

$$\left\| \int_{-\infty}^t \mathcal{U}(t-s) F(s) ds \right\|_{L^p(\mathbb{R}; L^q(C(\mathbb{S}_\rho^1)))} \lesssim \|F\|_{L^{\tilde{p}'}(\mathbb{R}; L^{\tilde{q}'}(C(\mathbb{S}_\rho^1)))},$$

once again including the case $(p, q) = (\infty, 2)$. A slight adjustment of these arguments implies that

$$\left\| \int_{-\infty}^t \sqrt{\Delta_g} \mathcal{U}(t-s) F(s) ds \right\|_{L^p(\mathbb{R}; L^q(C(\mathbb{S}_\rho^1)))} \lesssim \|F\|_{L^{\tilde{p}'}(\mathbb{R}; L^{\tilde{q}'}(C(\mathbb{S}_\rho^1)))}.$$

Corollary 1.3 now follows by scaling considerations and Littlewood-Paley theory as before. \square

It is now well-known via the standard TT^* argument (see for instance Keel and Tao [18], though the techniques originated much earlier) that if one has a dispersive estimate of the form

$$(3.12) \quad \left\| \beta \left(\sqrt{\Delta_g} \right) \mathcal{U}(t) g \right\|_{L^\infty(C(\mathbb{S}_\rho^1))} \lesssim (1+t)^{-\frac{1}{2}} \|g\|_{L^1(C(\mathbb{S}_\rho^1))}$$

for data g localized to unit frequency, then (3.9) will follow. The estimate (3.12) in turn follows by establishing bounds on the supremum of the Schwartz kernel of $\beta \left(\sqrt{\Delta_g} \right) \mathcal{U}(t)$. On \mathbb{R}^2 , this is typically accomplished by oscillatory integral methods. On flat cones, analogous oscillatory integrals appear to be very difficult to obtain. Instead, we work entirely on the spatial domain, treating the Littlewood-Paley cutoffs as operators which regularize the corresponding kernels. The dispersive estimates will then follow by showing that the regularized kernel is essentially bounded by its average over a set of unit size.

3.2. Bounds on the “geometric” term. We may unify the formulae in (1.18) and (1.20) to obtain a kernel which is supported in the union of Regions II and III. It will be written as the sum of two terms which we (somewhat) informally describe as a “geometric” term and a “diffractive” term,

$$(3.13) \quad K_{\mathcal{U}(t)}(t, r_1, \theta_1; r_2, \theta_2) = K_{\mathcal{U}(t)}^{\text{geom}}(t, r_1, \theta_1; r_2, \theta_2) + K_{\mathcal{U}(t)}^{\text{diff}}(t, r_1, \theta_1; r_2, \theta_2).$$

The expression in (1.18) and the first term in (1.20) can be unified to form the geometric term

$$(3.14) \quad K_{\mathcal{U}(t)}^{\text{geom}}(t, r_1, \theta_1; r_2, \theta_2) = \Psi(t, r_1, r_2, \theta_1 - \theta_2),$$

where Ψ is defined as

$$(3.15) \quad \Psi(t, r_1, r_2, \theta) \stackrel{\text{def}}{=} \sum_{-\pi \leq \theta + j \cdot 2\pi\rho \leq \pi} [t^2 - r_1^2 - r_2^2 + 2r_1r_2 \cos(\theta + j \cdot 2\pi\rho)]_+^{-\frac{1}{2}}.$$

Remark 3.5. The summation in (3.15) is dependent upon the relative location of the points in the integral kernel on the cone. However the total number of terms in the sum is no more than $1 + 1/\rho$. Also note that it vanishes for $|\theta| > \pi$.

The diffractive term will be the remaining term in (1.20); it is supported solely in Region III. To simplify our expression for it, we use the abbreviations

$$\begin{aligned} \alpha &\stackrel{\text{def}}{=} \frac{t^2 - r_1^2 - r_2^2}{2r_1r_2} = \frac{t^2 - (r_1 + r_2)^2}{2r_1r_2} + 1 & \beta &\stackrel{\text{def}}{=} \cosh^{-1}(\alpha) \\ \varphi_1 &\stackrel{\text{def}}{=} \frac{\pi + (\theta_1 - \theta_2)}{\rho} & \varphi_2 &\stackrel{\text{def}}{=} \frac{\pi - (\theta_1 - \theta_2)}{\rho}. \end{aligned}$$

We now write $K_{\mathcal{U}(t)}^{\text{diff}}(t, r_1, \theta_1; r_2, \theta_2)$ as

$$(3.16) \quad K_{\mathcal{U}(t)}^{\text{diff}}(t, r_1, \theta_1; r_2, \theta_2) = -\frac{\mathbf{1}_{(0,t)}(r_1 + r_2)}{4\pi^2\rho(2r_1r_2)^{\frac{1}{2}}} \times \int_0^\beta [\alpha - \cosh(s)]^{-\frac{1}{2}} \left[\frac{\sin(\varphi_1)}{\cosh(s/\rho) - \cos(\varphi_1)} + \frac{\sin(\varphi_2)}{\cosh(s/\rho) - \cos(\varphi_2)} \right] ds.$$

Let $K_{\beta(\sqrt{\Delta_g})}(r_3, \theta_3; r_2, \theta_2)$ denote the kernel of $\beta(\sqrt{\Delta_g})$. To show the dispersive estimate (3.12), we will establish the stronger bounds

$$(3.17) \quad \int_{C(\mathbb{S}_\rho^1)} \left| K_{\beta(\sqrt{\Delta_g})}(r_3, \theta_3; r_2, \theta_2) K_{\mathcal{U}(t)}^{\text{geom}}(t, r_1, \theta_1; r_2, \theta_2) \right| r_2 dr_2 d\theta_2 \lesssim \min\left(t, t^{-\frac{1}{2}}\right),$$

and

$$(3.18) \quad \int_{C(\mathbb{S}_\rho^1)} \left| K_{\beta(\sqrt{\Delta_g})}(r_3, \theta_3; r_2, \theta_2) K_{\mathcal{U}(t)}^{\text{diff}}(t, r_1, \theta_1; r_2, \theta_2) \right| r_2 dr_2 d\theta_2 \lesssim \min\left(t, t^{-\frac{1}{2}}\right).$$

Since the heat kernel $e^{-t\Delta_g}$ satisfies Gaussian upper bounds, as discussed in Section 3.1, we may use a theorem of Davies [12, Theorem 3.4.10] that provides bounds on the kernel of Schwartz class functions of the Laplacian. Namely, we obtain the following bound on $K_{\beta(\sqrt{\Delta_g})}(r_3, \theta_3; r_2, \theta_2)$, where we take N to be sufficiently large:

$$\left| K_{\beta(\sqrt{\Delta_g})}(r_3, \theta_3; r_2, \theta_2) \right| \lesssim (1 + d_g^2(r_3, \theta_3; r_2, \theta_2))^{-N}.$$

On $\mathbb{S}_\rho^1 \stackrel{\text{def}}{=} \mathbb{R}/2\pi\rho\mathbb{Z}$, we take coordinates $\theta \in (-\pi\rho, \pi\rho]$ and assume without loss of generality that $\theta_1 = 0$. Let $\varepsilon > 0$ be a small parameter such that $\min(\pi, \pi\rho)/\varepsilon$ is an integer. We tile the portion of the cone $(0, \infty)_r \times (-\min(\pi, \pi\rho), \min(\pi, \pi\rho)]_\theta$ into “polar rectangles”

$$(3.19) \quad R_{k,\ell} \stackrel{\text{def}}{=} (k\varepsilon, (k+1)\varepsilon]_r \times \left(\frac{\ell}{k}\varepsilon, \frac{\ell+1}{k}\varepsilon\right]_\theta,$$

where $k > 0$ and

$$-\frac{\min(\pi, \pi\rho) \cdot k}{\varepsilon} \leq \ell \leq \frac{\min(\pi, \pi\rho) \cdot k}{\varepsilon} - 1.$$

When $k = 0$, we take

$$(3.20) \quad R_{0,1} \stackrel{\text{def}}{=} (0, \varepsilon]_r \times (-\min(\pi, \pi\rho), 0]_\theta \quad \text{and} \quad R_{0,2} \stackrel{\text{def}}{=} (0, \varepsilon]_r \times (0, \min(\pi, \pi\rho)]_\theta.$$

Observe that the diameter of each of these rectangles is approximately ε . Indeed, if (r, θ) lies in one of these, then

$$(3.21) \quad \begin{aligned} d_g^2\left(r, \theta; k\varepsilon, \frac{\ell\varepsilon}{k}\right) &= r^2 + k^2\varepsilon^2 - 2rk\varepsilon \cos\left(\theta - \frac{\ell\varepsilon}{k}\right) \\ &= (r - k\varepsilon)^2 + \mathcal{O}\left(rk\varepsilon \cdot \frac{\varepsilon^2}{k^2}\right) \\ &\lesssim \varepsilon^2 \end{aligned}$$

since $\cos(\phi) = 1 + \mathcal{O}(\phi^2)$. Therefore, if $(r_2, \theta_2) \in R_{k,\ell}$ we have

$$(1 + d_g^2(r_3, \theta_3; r_2, \theta_2))^{-N} \lesssim \left(1 + d_g^2\left(r_3, \theta_3; k\varepsilon, \frac{\ell\varepsilon}{k}\right)\right)^{-N}.$$

We may now bound the left hand side of (3.17) by a constant multiple of

$$(3.22) \quad \sum_{k,\ell} \left(1 + d_g^2\left(r_3, \theta_3; k\varepsilon, \frac{\ell\varepsilon}{k}\right)\right)^{-N} \int_{R_{k,\ell}} \Psi(r_2, \theta_2; r_1, \theta_1) r_2 dr_2 d\theta_2.$$

Thus, if we can show that

$$(3.23) \quad \int_{R_{k,\ell}} \Psi(t, r_1, \theta_1; r_2, \theta_2) r_2 dr_2 d\theta_2 \lesssim \min(t, t^{-\frac{1}{2}}),$$

then (3.17) will follow because

$$(3.24) \quad \sum_{k,\ell} \left(1 + d_g^2 \left(r_3, \theta_3; k\varepsilon, \frac{\ell\varepsilon}{k} \right) \right)^{-N} \lesssim 1.$$

The remaining inequality (3.23) will be a consequence of the following lemma.

Lemma 3.6. *Let R be a polar rectangle of the form*

$$R = \left\{ (r, \theta) : r_0 < r \leq r_0 + \varepsilon, \quad |\theta - \theta_0| \leq \frac{\varepsilon}{r_0}, \quad \text{and } |\theta| \leq \pi \right\}$$

lying either in the upper or lower half plane in \mathbb{R}^2 , where $\varepsilon > 0$ is a sufficiently small parameter. Then for any $r_1 > 0$,

$$(3.25) \quad \iint_R (t^2 - r_1^2 - r^2 + 2r_1 r \cos(\theta))_+^{-\frac{1}{2}} r dr d\theta \lesssim \min(t, t^{-\frac{1}{2}}).$$

We postpone the proof of the lemma and show how it yields the estimate (3.23). Observe that for any rectangle defined above, either

$$R_{k,\ell} \subset (0, \infty) \times [0, \min(\pi, \pi\rho)] \quad \text{or} \quad R_{k,\ell} \subset (0, \infty) \times (-\min(\pi, \pi\rho), 0].$$

We now consider the natural identification of $(0, \infty) \times [0, \min(\pi, \pi\rho)]$ and $(0, \infty) \times (-\min(\pi, \pi\rho), 0]$ with a convex subset of either the upper half plane or the lower half plane. When $j = 0$ in (3.15), the desired estimate on the corresponding term follows directly from the lemma. Otherwise, we make a change of coordinates $\tilde{\theta} = \theta_2 + j \cdot 2\pi\rho$ and the condition $|\theta_2 + j \cdot 2\pi\rho| \leq \pi$ means that it is sufficient to assume that $|\tilde{\theta}| \leq \pi$ for any $(r, \tilde{\theta})$ in the translated rectangle $R_{k,\ell} + (0, j \cdot 2\pi\rho)$. Hence the lemma yields the desired estimate for other values of j .

Proof of Lemma 3.6. It suffices to treat the case where $0 \leq \theta \leq \pi$, as a slight adjustment of the arguments below will handle the remaining cases. We switch to polar coordinates $(\tilde{r}, \tilde{\theta})$ centered at $(r_1, 0)$, and let d_0 be the fixed distance

$$d_0 = d_g(r_1, 0; r_0, \theta_0).$$

By the law of cosines, $r_1^2 + r^2 - 2r_1 r \cos(\theta) = d_g^2(r_1, 0; r, \theta)$ is the square of the Euclidean distance from $(r_1, 0)$ to (r, θ) , and by (3.21), the diameter of R is $O(\varepsilon)$. Given these observations, there exists a uniform constant $C > 0$ and an angle $\tilde{\theta}_0$ such that $R \subset \tilde{R}$ where

$$\tilde{R} \stackrel{\text{def}}{=} \left\{ (\tilde{r}, \tilde{\theta}) : d_0 - C\varepsilon \leq \tilde{r} \leq d_0 + C\varepsilon \quad \text{and} \quad |\tilde{\theta} - \tilde{\theta}_0| \leq \frac{C\varepsilon}{d_0} \right\}.$$

Since the change of coordinates is an isometry, we have the following bounds on the integral in (3.25):

$$(3.26) \quad \iint_R [t^2 - d_g^2(r_1, 0; r, \theta)]_+^{-\frac{1}{2}} r dr d\theta \leq \iint_{\tilde{R}} (t^2 - \tilde{r}^2)_+^{-\frac{1}{2}} \tilde{r} d\tilde{r} d\tilde{\theta}.$$

When $t \geq 1$, we use that $(t + \tilde{r})^{-\frac{1}{2}} \leq t^{-\frac{1}{2}}$ and bound the integral on the right by

$$t^{-\frac{1}{2}} (d_0 + C\varepsilon) \int_{r_0}^{r_0 + \varepsilon} \int_{|\tilde{\theta} - \tilde{\theta}_0| \leq \frac{C\varepsilon}{d_0}} (t - \tilde{r})_+^{-\frac{1}{2}} d\tilde{\theta} d\tilde{r} \lesssim t^{-\frac{1}{2}} (d_0 + C\varepsilon) \cdot \frac{C\varepsilon}{d_0} \lesssim t^{-\frac{1}{2}}.$$

An straightforward adjustment of this computation handles the case where $d_0 \lesssim \varepsilon$. When $t \leq 1$, we can take the limits of integration in r so that they do not exceed t . Hence, the integral on the left in (3.26) can be bounded by a constant multiple of

$$t \int_0^\pi \int_0^t (t^2 - \tilde{r}^2)_+^{-\frac{1}{2}} d\tilde{r} d\tilde{\theta} = \frac{\pi^2 t}{2}.$$

This concludes the proof. \square

3.3. Bounds on the “diffractive” term. In order to handle the diffractive terms, we prove the following:

Lemma 3.7. *The kernel $K_{\mathcal{U}(t)}^{\text{diff}}$ defined in (3.16) satisfies the following pointwise bounds with an implicit constant independent of θ_1 and θ_2 :*

$$(3.27) \quad \left| K_{\mathcal{U}(t)}^{\text{diff}}(t, r_1, \theta_1; r_2, \theta_2) \right| \lesssim [t^2 - (r_1 + r_2)^2]_+^{-\frac{1}{2}}.$$

Given this lemma, a straightforward adaptation of the proof of Lemma 3.6 shows that

$$\iint_{R_{k,\ell}} \left| K_{\mathcal{U}(t)}^{\text{diff}}(t, r_1, \theta_1; r_2, \theta_2) \right| r_1 dr_1 d\theta_1 \lesssim \min(t, t^{-\frac{1}{2}});$$

indeed, the proof of this fact is simpler since coordinates need not be shifted away from the cone tip. The desired dispersive estimate will then follow from the reasoning preceding Lemma 3.6. The only difference here is that the tiling of polar rectangles must now occur over $(0, \infty)_r \times (-\pi\rho, \pi\rho]_\theta$.

Proof of Lemma 3.7. Recall the notation

$$\begin{aligned} \alpha &\stackrel{\text{def}}{=} \frac{t^2 - r_1^2 - r_2^2}{2r_1 r_2} = \frac{t^2 - (r_1 + r_2)^2}{2r_1 r_2} + 1 & \beta &\stackrel{\text{def}}{=} \cosh^{-1}(\alpha) \\ \varphi_1 &\stackrel{\text{def}}{=} \frac{\pi + (\theta_1 - \theta_2)}{\rho} & \varphi_2 &\stackrel{\text{def}}{=} \frac{\pi - (\theta_1 - \theta_2)}{\rho}. \end{aligned}$$

Given the factor $(2r_1 r_2)^{-\frac{1}{2}}$ in (3.16), it suffices to show for $\varphi = \varphi_1, \varphi_2$ that

$$(3.28) \quad \left| \int_0^\beta [\alpha - \cosh(s)]^{-\frac{1}{2}} \left[\frac{\sin(\varphi)}{\cosh(s/\rho) - \cos(\varphi)} \right] ds \right| \lesssim (\alpha - 1)^{-\frac{1}{2}}.$$

We will treat the cases $\beta \geq 1$ and $\beta \leq 1$ separately. In the first case, we will approximate $\beta = \cosh^{-1}(\alpha) \approx \log(2\alpha)$. In the second case, we will approximate $\beta \approx \sqrt{2(\alpha - 1)}$.

We begin with the case where $\beta \geq 1$, which in turn implies that $\alpha \geq \cosh(1) > \frac{3}{2}$. We will choose a constant $M = M(\beta) \in [\frac{1}{2}, \beta)$ and write the domain of integration in (3.28) as $[0, \beta) = [0, M) \sqcup [M, \beta)$. The constant M will be chosen so that the following three conditions are satisfied. First, we want a small uniform constant $\delta > 0$ so that for $s \in [0, M]$,

$$(3.29) \quad \cosh(s) - 1 \leq (1 - \delta)(\alpha - 1) = (1 - \delta) \left(\frac{t^2 - (r_1 + r_2)^2}{2r_1 r_2} \right).$$

For $s \in [M, \beta)$, we linearize $\cosh(s)$:

$$(3.30) \quad \cosh(s) = \cosh(\beta) + (s - \beta) \sinh(\beta) + R_1(s, \beta).$$

The second condition is to ensure that M is sufficiently large so that for $s \in [M, \beta]$,

$$(3.31) \quad |R_1(s, \beta)| \leq \frac{1}{2}(\beta - s) \sinh(\beta).$$

Finally, we want to see that M is bounded from below:

$$(3.32) \quad M \geq \max\left(\beta - 1, \frac{1}{2}\right).$$

Assuming that this choice of M can be made, we now turn to the integral in (3.28). We first treat the integral over $[0, M]$. Given (3.29), we have that for small φ (cp. [9, (4.15)])

$$(3.33) \quad \int_0^M [\alpha - \cosh(s)]^{-\frac{1}{2}} \left[\frac{|\sin(\varphi)|}{\cosh(s/\rho) - \cos(\varphi)} \right] ds \lesssim (\alpha - 1)^{-\frac{1}{2}} \int_0^M \frac{|\varphi|}{\varphi^2 + s^2} ds.$$

A similar argument works for φ in neighborhoods of integral multiples of 2π . Otherwise, stronger estimates hold since the denominator in brackets is bounded away from 0. Hence, this term is bounded by $(\alpha - 1)^{-\frac{1}{2}}$. Turning to the integral over $[M, \beta]$, the approximations (3.30) and (3.31) imply that

$$\alpha - \cosh(s) = (\beta - s) \sinh(\beta) - R_1(s, \beta) \geq \frac{1}{2}(\beta - s) \sinh(\beta).$$

Furthermore, choosing $M \geq \frac{1}{2}$ ensures that the denominator in brackets can be uniformly bounded from below when $s \in [M, \beta]$. Thus, for any φ we see

$$(3.34) \quad \int_M^\beta [\alpha - \cosh(s)]^{-\frac{1}{2}} \left[\frac{|\sin(\varphi)|}{\cosh(s/\rho) - \cos(\varphi)} \right] ds \lesssim \int_M^\beta \sinh(\beta)^{-\frac{1}{2}} (\beta - s)^{-\frac{1}{2}} ds.$$

The inequality $M \geq \beta - 1$ in (3.32) ensures that the integral on the right is bounded by a constant multiple of

$$\sinh(\beta)^{-\frac{1}{2}} = [\cosh(\beta) - e^{-\beta}]^{-\frac{1}{2}} \lesssim (\alpha - 1)^{-\frac{1}{2}},$$

with the inequality following from the fact that $\beta \geq 1$.

We now show that choosing $M = \max(\beta - \tanh(\beta), \frac{1}{2})$ satisfies the requisite estimates (3.29), (3.31), and (3.32). The last property follows easily from the fact that $\tanh(\beta) \leq 1$. The remainder estimate (3.31) also follows easily, for any $s \in [M, \beta]$ will satisfy $\beta - s \leq \tanh(\beta)$, and hence

$$|R_1(s, \beta)| \leq \frac{1}{2}(\beta - s)^2 \cosh(\beta) \leq \frac{1}{2}(\beta - s) \cosh(\beta) \tanh(\beta) \leq \frac{1}{2}(\beta - s) \sinh(\beta).$$

All that remains is to check that with this choice of M there exists $\delta > 0$ so that (3.29) is satisfied. Observe that for $\beta \geq 1$, $e^\beta < 2 \cosh(\beta) = 2\alpha \leq 2e^\beta$. Therefore, if $M = \beta - \tanh(\beta)$ and $s \in [0, M]$, then

$$\begin{aligned} \cosh(s) - 1 &\leq \exp[\beta - \tanh(\beta)] - 1 \\ &\leq 2\alpha \exp[-\tanh(1)] - 1 \\ &\leq \alpha \exp[\log(2) - \tanh(1)] - 1. \end{aligned}$$

Since $\log(2) - \tanh(1) < 0$, (3.29) will then hold for any $\delta > 0$ satisfying $1 - \delta > \exp[\log(2) - \tanh(1)]$. If $M = \frac{1}{2}$, then since $\cosh(1) \leq \cosh(\beta) = \alpha$, we have

$$\cosh\left(\frac{1}{2}\right) - 1 \leq \frac{\cosh(\frac{1}{2})}{\cosh(1)} \alpha - 1.$$

Similarly, (3.29) will hold provided $1 - \delta > \frac{\cosh(1/2)}{\cosh(1)}$.

We now turn to the case $\beta \leq 1$ (equivalently $\alpha \leq \cosh(1)$). Using the inverse function theorem, we observe β^2 is differentiable in α when $\alpha > 1$, and

$$\frac{d}{d\alpha}\beta^2 = 2\beta \frac{d\beta}{d\alpha} = \frac{2\beta}{\sinh(\beta)}.$$

By l'Hôpital's rule, this function is bounded from below as $\beta \rightarrow 0$. Furthermore, $\frac{2\beta}{\sinh(\beta)}$ is a decreasing function for $\beta \geq 0$. The fundamental theorem of calculus thus allows us to conclude that for any $\alpha \leq \cosh(1)$, we have $\beta^2 \geq \frac{2}{\sinh(1)}(\alpha - 1) > 1.7(\alpha - 1)$. Since $\cosh(\beta) \geq 1 + \frac{\beta^2}{2}$, we have the upper and lower bounds

$$(3.35) \quad \sqrt{2(\alpha - 1)} \geq \beta \geq \sqrt{1.7(\alpha - 1)}.$$

As before, we will select $M = M(\beta) \in (0, \beta)$ and estimate the integral in (3.16) by partitioning $\int_0^\beta = \int_0^M + \int_M^\beta$. We begin by insisting that for $s \in [0, M]$,

$$(3.36) \quad \cosh(s) - 1 \leq \frac{1}{2}(\alpha - 1).$$

We also require that for $s \in [M, \beta]$, the remainder estimate (3.31) in the linear approximation is satisfied. Finally, we insist that $M \geq \frac{\beta}{2}$. If these conditions hold, then (3.36) ensures that the integral over $[0, M]$ can be estimated as in (3.33). However, this time we need a different argument to estimate the integral on the left in (3.34). For small φ , we have

$$(3.37) \quad \begin{aligned} & \int_M^\beta [\alpha - \cosh(s)]^{-\frac{1}{2}} \left[\frac{|\sin(\varphi)|}{\cosh(s/\rho) - \cos(\varphi)} \right] ds \\ & \lesssim \int_M^\beta [(\beta - s) \sinh(\beta)]^{-\frac{1}{2}} \left[\frac{|\sin(\varphi)|}{\cosh(M/\rho) - \cos(\varphi)} \right] ds \\ & \lesssim \frac{|\varphi|}{\varphi^2 + \beta^2} \sinh(\beta)^{-\frac{1}{2}} \int_{\frac{\beta}{2}}^\beta (\beta - s)^{-\frac{1}{2}} ds \lesssim \frac{1}{\beta} \left(\frac{\beta}{\sinh(\beta)} \right)^{\frac{1}{2}}. \end{aligned}$$

We can now conclude that

$$\frac{1}{\beta} \left(\frac{\beta}{\sinh(\beta)} \right)^{\frac{1}{2}} \lesssim (\alpha - 1)^{-\frac{1}{2}}$$

by (3.35).

This time, we select $M = \sqrt{\frac{1}{2}(\alpha - 1)}$ and observe that $M \geq \frac{\beta}{2}$ by (3.35). Note that this choice of M implies (3.36) is satisfied. Indeed, by Taylor remainder estimates we have that when $|s| \leq 1$,

$$\left| \cosh(s) - 1 - \frac{s^2}{2} \right| \leq s^4 \frac{\cosh(1)}{4!},$$

which in turn implies that for $s \in [0, M]$

$$|\cosh(s) - 1| \leq s^2 \leq \frac{1}{2}(\alpha - 1).$$

To see that the remainder estimate (3.31) is satisfied here for $s \in [M, \beta]$, we will show that

$$\beta - \tanh(\beta) \leq \frac{\beta}{2} \leq \sqrt{\frac{1}{2}(\alpha - 1)} = M.$$

The second inequality follows from (3.35), so it suffices to prove the first. This is equivalent to showing that

$$\frac{1}{2}\beta - \tanh(\beta) \leq 0.$$

As a function of $\beta \geq 0$, $\frac{1}{2}\beta - \tanh(\beta)$ is convex and decreasing near 0. Therefore, the desired inequality holds for $\beta \in [0, 1]$ since $\tanh(1) > \frac{1}{2}$. \square

4. APPLICATIONS

In this section, we present some applications of the above analysis. We begin by discussing Morawetz estimates on Euclidean cones and, more generally, metric cones. We then present Strichartz estimates for the appropriate wave equation IVPs on wedge domains, polygonal domains, and Euclidean surfaces with conic singularities. We close by discussing well-posedness results for nonlinear wave equations in these settings.

4.1. Morawetz Estimates on the Euclidean Cone. Following ideas in [5] and [21], we prove Morawetz, or local energy decay, estimates for the wave equation on Euclidean cones. To begin, we define the multiplier operator

$$(\Omega^s \phi)(r, \theta) = r^s \phi(r, \theta).$$

Recall from Section 2.2 that our definition of the homogeneous Sobolev spaces $\dot{H}^s(C(\mathbb{S}_\rho^1))$ is in terms of the spectral decomposition of the Laplace-Beltrami operator Δ_g . We may thus define the following subspace,

$$(4.1) \quad \dot{H}_{\geq m}^s(C(\mathbb{S}_\rho^1)) \stackrel{\text{def}}{=} \left\{ f \in \dot{H}^s(C(\mathbb{S}_\rho^1)) : \Pi_j f = 0 \text{ for } \nu_j < \nu_m \right\},$$

where Π_j is the spectral projector defined in (2.9) and ν_m is the m -th modified eigenvalue of Δ_g . We have the following theorem.

Theorem 4.1. *Let $m \geq 1$ be an integer and $0 < \alpha < \frac{1}{4} + \frac{1}{2}\nu_m$. Given a solution u to the wave equation IVP (1.1), there exists a constant $C = C(m, \alpha)$ such that for all $f \in \dot{H}_{\geq m}^{\frac{1}{2}}(C(\mathbb{S}_\rho^1))$ and $g \in \dot{H}_{\geq m}^{-\frac{1}{2}}(C(\mathbb{S}_\rho^1))$, we have*

$$(4.2) \quad \left\| \Omega^{-\frac{1}{2}-2\alpha} \Delta_g^{\frac{1}{4}-\alpha} u \right\|_{L^2(\mathbb{R} \times C(\mathbb{S}_\rho^1))} \leq C \left(\|f\|_{\dot{H}_{\geq m}^{\frac{1}{2}}(C(\mathbb{S}_\rho^1))} + \|g\|_{\dot{H}_{\geq m}^{-\frac{1}{2}}(C(\mathbb{S}_\rho^1))} \right).$$

Remark 4.2. The proof may be performed on each spherical harmonic separately; this follows directly from the Hankel function analysis in [5]. We note that the proof involves only analysis on the radial operator, so the Morawetz estimate will hold in general for any metric cone $C(Y)$ as described in Section 2.1. In particular, we note that $\nu_0 > 0$ when $\dim(Y) > 1$, removing any restrictions on the bottom of the spectrum

The proof of Theorem 4.1 relies heavily on the results in [21] and [5]. By using the projections Π_j defined in (2.9) and orthogonality, it suffices to consider u , f , and g in the range of Π_j for some j . As a consequence, we may assume that all three functions depend only on t and r and that

$$\begin{cases} (D_t^2 - A_\nu) u(t, r) = 0 \\ u(0, r) = f(r) \\ \partial_t u(0, r) = g(r), \end{cases}$$

where $\nu^2 = \nu_j^2$, the j -th (modified) eigenvalue of $\Delta_{\mathbb{S}_\rho^1}$, and

$$A_\nu \stackrel{\text{def}}{=} -\partial_r^2 - \frac{1}{r}\partial_r + \frac{\nu^2}{r^2}.$$

We may then define powers of A_ν by

$$A_\nu^{\frac{\sigma}{2}} \stackrel{\text{def}}{=} \mathcal{H}_\nu \circ \Omega^\sigma \circ \mathcal{H}_\nu.$$

In [5], it is shown that the desired estimate (4.2) reduces to showing

$$(4.3) \quad \left\| \Omega^{-\frac{1}{2}-2\alpha} A_\nu^{\frac{1}{4}-\alpha} u \right\|_{L^2(\mathbb{R} \times C(\mathbb{S}_\rho^1))} \lesssim \left\| A_\nu^{\frac{1}{4}} f \right\|_{L^2(C(\mathbb{S}_\rho^1))} + \left\| A_\nu^{-\frac{1}{4}} g \right\|_{L^2(C(\mathbb{S}_\rho^1))}.$$

Observe that

$$(\mathcal{H}_\nu u)(t, \lambda) = \cos(t\lambda)(\mathcal{H}_\nu f)(\lambda) + \frac{\sin(t\lambda)}{\lambda}(\mathcal{H}_\nu g)(\lambda).$$

Taking the Fourier transform of u in time yields

$$(\mathcal{F}_t \mathcal{H}_\nu u)(\tau, \lambda) = \frac{1}{\sqrt{\lambda}}(\delta(\tau + \lambda)h_+(\lambda) + \delta(\tau - \lambda)h_-(\lambda)),$$

where

$$h_\pm(\lambda) = \frac{1}{2} \left(\sqrt{\lambda}(\mathcal{H}_\nu f)(\lambda) \pm \frac{1}{i\sqrt{\lambda}}(\mathcal{H}_\nu g)(\lambda) \right).$$

Moreover,

$$\|h_\pm\|_{L^2(\mathbb{R}_+; \lambda d\lambda)} \lesssim \left\| A_\nu^{\frac{1}{4}} f \right\|_{L^2(C(\mathbb{S}_\rho^1))} + \left\| A_\nu^{-\frac{1}{4}} g \right\|_{L^2(C(\mathbb{S}_\rho^1))}.$$

For $\tau > 0$,

$$\left(A_\nu^{-\frac{1}{4}-\alpha} \Omega^{\frac{1}{2}-2\alpha} \mathcal{F}_t \mathcal{H}_\nu u \right)(\tau, \lambda) = \tau^{1-2\alpha} k_{\nu, \nu}(\lambda, \tau)^{-\frac{1}{2}-2\alpha} h_-(\tau),$$

where $k_{\nu, \nu}$ is the integral kernel of $A_\nu^{\frac{\sigma}{2}}$ as in [5] and related to that in (2.5). Correspondingly, there is an expression for $\tau < 0$ involving h_+ . The inequality (4.3) then follows from the calculation in [5] which establishes that

$$\left\| A_\nu^{-\frac{1}{4}-\alpha} \Omega^{\frac{1}{2}-2\alpha} \mathcal{F}_t \mathcal{H}_\nu u \right\|_{L^2(\mathbb{R}_\pm \times \mathbb{R}_+; \lambda d\tau d\lambda)} = C_{\nu, \alpha} \|h_\mp\|_{L^2(\mathbb{R}_+; \lambda d\lambda)}$$

for some constant $C_{\nu, \alpha}$ uniformly bounded in ν .

4.2. Global Strichartz estimates on wedge domains. Let Ω be a planar domain of the form

$$\Omega = \{(r, \theta) : r > 0 \text{ and } 0 \leq \theta \leq \alpha\} \subseteq \mathbb{R}^2,$$

where (r, θ) denote standard polar coordinates and $\alpha \leq 2\pi$. We consider solutions to the inhomogeneous IBVP for the wave equation,

$$(4.4) \quad \begin{cases} (D_t^2 - \Delta) u(t, r, \theta) = F(t, r, \theta) \\ u(0, r, \theta) = f(r, \theta) \\ \partial_t u(0, r, \theta) = g(r, \theta), \end{cases}$$

satisfying either Dirichlet or Neumann homogeneous boundary conditions, i.e.

$$(4.5) \quad u|_{\mathbb{R} \times \partial\Omega} \equiv 0 \quad \text{or} \quad \partial_n u|_{\mathbb{R} \times \partial\Omega} \equiv 0.$$

Here, ∂_n denotes the normal derivative along the boundary. When Dirichlet conditions are imposed, we take Δ to mean the Friedrichs extension of the Laplacian

$D_{x_1}^2 + D_{x_2}^2$ acting on $\mathcal{C}_c^\infty(\Omega)$. If Neumann conditions are imposed, we take Δ to mean the Friedrichs extension of the same differential operator acting on smooth functions which vanish in a neighborhood of the vertices and whose normal derivative vanishes on $\partial\Omega$.

In this section, we discuss global Strichartz on these domains. Namely, we prove the following theorem.

Theorem 4.3. *Suppose u is a solution to the wave equation IBVP (4.4)-(4.5) with initial data $(f, g) \in \dot{H}^\gamma(\Omega) \times \dot{H}^{\gamma-1}(\Omega)$. Then for any triple (p, q, γ) satisfying the hypotheses of Theorem 1.2, u satisfies the estimates*

$$(4.6) \quad \|u\|_{L^p(\mathbb{R}; L^q(\Omega))} + \|(u, \partial_t u)\|_{L^\infty(\mathbb{R}; \dot{H}^\gamma(\Omega) \times \dot{H}^{\gamma-1}(\Omega))} \\ \lesssim \|(f, g)\|_{\dot{H}^\gamma(\Omega) \times \dot{H}^{\gamma-1}(\Omega)} + \|F\|_{L^{\bar{p}'}(\mathbb{R}; L^{\bar{q}'}(\Omega))},$$

Recently, there has been some partial progress in proving such bounds for domains with smooth boundaries; see [22], [4], and [3]. However, the examples of Ivanovici [17] show that when the smooth boundary possesses a point of convexity, the full range of scale-invariant Strichartz estimates cannot hold. In contrast, we will obtain the full range of Strichartz estimates, even for convex domains. The fact that the boundaries we consider are flat, except at the corners, seems to limit the problems created by multiply-reflected rays. However, as is implicit in Section 3, the diffractive effects created by interaction with the corners create their own challenges.

We note that the diffraction occurring from the tip of a wedge was first analyzed by Sommerfeld in [24] via special function computations. For a nice treatment, we recommend the book by Stakgold [25, Chapter 7.12], who uses the method of reflection to describe Green's function for the Helmholtz operator on a wedge.

Proof of Theorem 4.3. By density arguments similar to those in Section 2.2, it suffices to assume that $u(t, \cdot)$, f , and g are all smooth and compactly supported in $\bar{\Omega}$. Furthermore, by taking Fourier series in θ , we may assume that $u(t, r, \theta)$ may be written as either

$$(4.7) \quad \frac{1}{\sqrt{\alpha}} \sum_{j=1}^{\infty} u_j(t, r) \sin\left(\frac{j\pi\theta}{\alpha}\right) \quad \text{or} \quad u_0(t, r) + \frac{1}{\sqrt{\alpha}} \sum_{j=1}^{\infty} u_j(t, r) \cos\left(\frac{j\pi\theta}{\alpha}\right),$$

depending on whether Dirichlet or Neumann boundary conditions are taken. The separation of variables approach in Section 2 can now be applied to these expansions, giving rise to a spectral resolution of the Dirichlet or Neumann Laplacian. We may then use this spectral resolution to define homogeneous Sobolev spaces on Ω .

Given (4.7), $u(t, r, \theta)$ can be extended to a function on $C(\mathbb{S}_\rho^1)$ for $\rho = \frac{\alpha}{\pi}$. In addition, $L^2(C(\mathbb{S}_\rho^1))$ may be decomposed as a sum of the L^2 -orthogonal subspaces $L_{\text{Dir}}^2(C(\mathbb{S}_\rho^1))$ and $L_{\text{Neu}}^2(C(\mathbb{S}_\rho^1))$. The subspace $L_{\text{Dir}}^2(C(\mathbb{S}_\rho^1))$ is defined as all functions $a \in L^2(C(\mathbb{S}_\rho^1))$ which may be written

$$(4.8) \quad a(r, \theta) = \frac{1}{\sqrt{\alpha}} \sum_{j=1}^{\infty} a_j(r) \sin\left(\frac{j\pi\theta}{\alpha}\right).$$

Similarly, $L_{\text{Neu}}^2(C(\mathbb{S}_\rho^1))$ is defined as all functions which can be written as

$$(4.9) \quad a(r, \theta) = a_0(r) + \frac{1}{\sqrt{\alpha}} \sum_{j=1}^{\infty} a_j(r) \cos\left(\frac{j\pi\theta}{\alpha}\right).$$

It is not difficult to see that $L_{\text{Dir}}^2(C(\mathbb{S}_\rho^1)) \cap \text{Dom}(\Delta_g)$ and $L_{\text{Neu}}^2(C(\mathbb{S}_\rho^1)) \cap \text{Dom}(\Delta_g)$ are invariant under the action of Δ_g . Therefore, solutions $u(t, \cdot)$ to the Dirichlet and Neumann problems extend naturally to functions in $L_{\text{Dir}}^2(C(\mathbb{S}_\rho^1))$ and $L_{\text{Neu}}^2(C(\mathbb{S}_\rho^1))$ respectively. Furthermore, there is a natural identification between the action of $\Delta^{\frac{\alpha}{2}}$ on $L^2(\Omega)$ and the action of $\Delta_g^{\frac{\alpha}{2}}$ on $L_{\text{Dir}}^2(C(\mathbb{S}_\rho^1))$ and $L_{\text{Neu}}^2(C(\mathbb{S}_\rho^1))$. As such,

$$\|f\|_{\dot{H}^\gamma(\Omega)} \approx \|f\|_{\dot{H}^\gamma(C(\mathbb{S}_\rho^1))},$$

and similarly for g . Thus, by Theorem 1.2 we have that

$$(4.10) \quad \|u\|_{L^p(\mathbb{R}; L^q(\Omega))} \approx \|u\|_{L^p([-T, T]; L^q(C(\mathbb{S}_\rho^1)))} \lesssim \|f\|_{\dot{H}^\gamma(C(\mathbb{S}_\rho^1))} + \|g\|_{\dot{H}^{\gamma-1}(C(\mathbb{S}_\rho^1))}.$$

This concludes the proof. \square

4.3. Polygonal Domains and Euclidean Surfaces with Conic Singularities.

In this section, we let Ω be an open domain in the plane with a piecewise linear boundary (i.e. a polygonal domain), and we consider local Strichartz estimates for solutions to the IBVP (4.4)-(4.5). We prove the following theorem.

Theorem 4.4. *Suppose u is a solution to the wave equation IBVP (4.4)-(4.5) with initial data $(f, g) \in H^\gamma(\Omega) \times H^{\gamma-1}(\Omega)$. Then for any triple (p, q, γ) satisfying the hypotheses of Theorem 1.2, u satisfies the estimates*

$$(4.11) \quad \|u\|_{L^p([-T, T]; L^q(\Omega))} + \|(u, \partial_t u)\|_{L^\infty([-T, T]; H^\gamma(\Omega) \times H^{\gamma-1}(\Omega))} \\ \lesssim \|(f, g)\|_{H^\gamma(\Omega) \times H^{\gamma-1}(\Omega)} + \|F\|_{L^{\tilde{p}'}([-T, T]; L^{\tilde{q}'}(\Omega))},$$

Proof. We begin by observing that it suffices to show this for sufficiently small T and smooth initial data (f, g) . By finite propagation speed, this in turn allows us to assume that the solution is compactly supported in a set that is either isometric to a subset of the upper half plane or isometric to a corner of angle α . The most difficult case here is the latter; the former case will follow by Strichartz estimates on the plane and Sobolev space estimates analogous to those below. The arguments here will also apply to any Euclidean surface with conic singularities, as seen in the Schrödinger equation case in [2].

We now assume that f and g are supported in a set of the form $\{(r, \theta) : 0 < r < \delta \text{ and } 0 \leq \theta \leq \alpha\}$. As before, we expand $u(t, r, \theta)$ in the form (4.7) and observe that u extends to a function on the cone $C(\mathbb{S}_\rho^1)$ with $\rho = \frac{\alpha}{\pi}$. For each time t , $u(t, \cdot)$ lies in one of the spaces $L_{\text{Dir}}^2(C(\mathbb{S}_\rho^1))$ or $L_{\text{Neu}}^2(C(\mathbb{S}_\rho^1))$ defined above. Reasoning analogously to (4.10), Theorem 4.4 is a consequence of Corollary 1.4 and the following lemma. \square

Lemma 4.5. *Let $a(r, \theta)$ be a smooth function on a wedge of the form $\{(r, \theta) : \delta_0 < r < \delta \text{ and } 0 \leq \theta \leq \alpha\}$ that can be written in one of the forms (4.8) or (4.9). Then for any $s \in [-2, 2]$, there exists a constant C independent of δ_0 such that*

$$(4.12) \quad \|a\|_{H^s(C(\mathbb{S}_\rho^1))} \leq C \|a\|_{H^s(\Omega)},$$

where $\|a\|_{H^s(C(\mathbb{S}_\rho^1))}$ denotes the quantity formed by extending the function to $C(\mathbb{S}_\rho^1)$, $\rho = \frac{\alpha}{\pi}$, and taking the corresponding Sobolev norm.

Proof. When $s = 2$, we have that

$$\|f\|_{H^2(C(\mathbb{S}_\rho^1))} \approx \sum_{k=0}^2 \|\nabla_g^k f\|_{L^2(C(\mathbb{S}_\rho^1))} \approx \sum_{k=0}^2 \|\nabla_g^k f\|_{L^2(\Omega)} \approx \|f\|_{H^2(\Omega)}.$$

By interpolation with $s = 0$, (4.12) holds for any $s \in [0, 2]$. To handle the negative Sobolev indices, we let Π_{Dir} and Π_{Neu} denote the corresponding orthogonal projections onto $L_{\text{Dir}}^2(C(\mathbb{S}_\rho^1))$ and $L_{\text{Neu}}^2(C(\mathbb{S}_\rho^1))$ respectively. Now let $\chi = \chi(r)$ be a compactly supported smooth function on $C(\mathbb{S}_\rho^1)$ satisfying $\chi \equiv 1$ for $0 \leq r \leq \delta$. We then have that if a can be expanded in terms of the sine basis, then its extension to the cone satisfies $\Pi_{\text{Dir}} a = a$, and hence

$$\begin{aligned} \|a\|_{H^{-2}(C(\mathbb{S}_\rho^1))} &= \sup\{|\langle a, b \rangle_{L^2(C(\mathbb{S}_\rho^1))}| : \|b\|_{H^2(C(\mathbb{S}_\rho^1))} = 1\} \\ &= 2 \sup\{|\langle a, \chi \Pi_D b \rangle_{L^2(\Omega)}| : \|b\|_{H^2(C(\mathbb{S}_\rho^1))} = 1\} \\ &\lesssim \|a\|_{H^{-2}(\Omega)} \|\chi \Pi_D b\|_{H^2(\Omega)} \lesssim \|a\|_{H^{-2}(\Omega)}. \end{aligned}$$

Here, the last inequality follows from

$$\|\chi \Pi_{\text{Dir}} b\|_{H^2(\Omega)} \lesssim \|\chi \Pi_{\text{Dir}} b\|_{H^2(C(\mathbb{S}_\rho^1))} \lesssim \|\Pi_{\text{Dir}} b\|_{H^2(C(\mathbb{S}_\rho^1))} \leq \|b\|_{H^2(C(\mathbb{S}_\rho^1))}.$$

When a can be expanded in terms of the cosine basis, the same proof applies with Π_{Dir} replaced by Π_{Neu} to show (4.12) when $s = -2$. Interpolation with $s = 0$ now completes the proof. \square

4.4. Well-posedness for nonlinear waves. As an application of the Strichartz estimates above, we note that these inequalities can be used to show that a theorem of Lindblad and Sogge [19] on \mathbb{R}^2 carries over to any of the contexts above (wedge domains, polygonal domains, Euclidean cones, and ESCSs). Specifically, we let X denote any one of these manifolds, imposing Dirichlet or Neumann boundary conditions as appropriate. Consider the semilinear initial value problem

$$(4.13) \quad \begin{cases} (D_t^2 - \Delta_g) v(t, x) = \pm |v|^{\kappa-1} v \\ v(0, x) = v_0(x) \in H^\gamma(X) \\ \partial_t v(0, x) = v_1(x) \in H^{\gamma-1}(X), \end{cases}$$

where $\gamma = \gamma(\kappa) = \max\left(\frac{3}{4} - \frac{1}{\kappa-1}, 1 - \frac{2}{\kappa-1}\right)$, i.e.

$$\gamma = \gamma(\kappa) = \begin{cases} \frac{3}{4} - \frac{1}{\kappa-1} & 3 < \kappa \leq 5 \\ 1 - \frac{2}{\kappa-1} & 5 \leq \kappa < \infty. \end{cases}$$

In general, the Sobolev index $\gamma(\kappa)$ is the lowest degree of regularity for which the problem (4.13) is locally well-posed. Indeed, Lindblad-Sogge [19] constructed explicit examples of solutions to the focusing problem on \mathbb{R}^2 that demonstrate it is ill-posed in Sobolev spaces below $\gamma(\kappa)$. For defocusing nonlinearities, Christ-Colliander-Tao [10] produced examples on \mathbb{R}^2 which show ill-posedness when the Sobolev regularity is below $1 - \frac{2}{\kappa-1}$. The well-posedness of the problem (4.13) is one of many important applications of Strichartz estimates. It is an example of how these inequalities efficiently handle the perturbative theory for these equations.

Theorem 4.6. *Suppose X is a 2-dimensional manifold where the local Strichartz estimates (4.11) are valid. Then given any pair of initial data in $(v_0, v_1) \in H^\gamma(X) \times$*

$H^{\gamma-1}(X)$ there exists $T > 0$ and a unique solution of (4.13) satisfying

$$(v(t, \cdot), \partial_t v(t, \cdot)) \in \mathcal{C}^0([0, T]; H^\gamma(X) \times H^{\gamma-1}(X)) \cap L^p([0, T]; L^{\frac{3}{2}(\kappa-1)}(X)),$$

where $p = \max\left(\frac{3}{\gamma(\kappa)}, \frac{3}{2}(\kappa-1)\right)$. Furthermore, if T^* denotes the maximal lifespan of the solution in $H^\gamma(X) \times H^{\gamma-1}(X)$, then either $T^* = \infty$ or

$$\|v\|_{L^{\frac{3}{2}(\kappa-1)}([0, T^*) \times X)} = \infty.$$

When global Strichartz estimates are available, we also have global existence when the initial data is sufficiently small in $\dot{H}^\gamma(X) \times \dot{H}^{\gamma-1}(X)$.

Corollary 4.7. *Suppose X is a 2-dimensional manifold where the global Strichartz estimates (4.6) are valid. Then the above theorem holds with inhomogeneous Sobolev spaces replaced by homogenous ones. Moreover, there exists $\varepsilon(\kappa) > 0$ for which the solution will exist globally in $\dot{H}^\gamma(X) \times \dot{H}^{\gamma-1}(X)$ (i.e. $T^* = \infty$) whenever the initial data satisfies*

$$\|f\|_{\dot{H}^\gamma(X)} + \|g\|_{\dot{H}^{\gamma-1}(X)} \leq \varepsilon(\kappa).$$

The proofs of these results are essentially due to Lindblad-Sogge [19]. The only complication is that the so-called “fractional Leibniz rule” is not apparent in our context, so we must take additional care regarding the function spaces we use to obtain a contraction. In particular, we make use of the estimate

$$(4.14) \quad \|u\|_{L^p([0, T]; L^q(X))} + \|(u, \partial_t u)\|_{L^\infty([0, T]; H^\gamma(X) \times H^{\gamma-1}(X))} \\ \lesssim \|(f, g)\|_{H^\gamma(X) \times H^{\gamma-1}(X)} + \|F\|_{L^{\frac{3}{2+\gamma}}([0, T]; L^{\frac{6}{7-4\gamma}}(X))}.$$

where p and q will satisfy (1.4) and $\frac{1}{p} + \frac{2}{q} = 1 - \gamma$. When $3 < \kappa < 9$, we may take $p = \frac{3}{\gamma}$ and $q = \frac{6}{3-4\gamma}$ as in Lemmas 4.1 and 4.2 of [19] to obtain local existence of the solution. When $9 \leq \kappa < \infty$, we instead take $p = \frac{3\kappa}{2+\gamma}$ and $q = \frac{6\kappa}{7-4\gamma}$, and local existence follows from a slight adjustment of Lemma 3.6 in [23].

The remainder of the theorem now follows by the same considerations as in Theorems 5.1 and 5.2 in [19] once it is seen that $v \in L^{\frac{3}{2}(\kappa-1)}([0, T] \times X)$. When $k \geq 9$, Hölder’s inequality gives

$$\|v\|_{L^{\frac{3}{2}(\kappa-1)}([0, T] \times X)} \leq \|v\|_{L^{\frac{3\kappa}{2+\gamma}}([0, T]; L^{\frac{6\kappa}{7-4\gamma}}(X))}^\theta \|v\|_{L^\infty([0, T]; L^{\frac{2}{1-\gamma}}(X))}^{1-\theta}$$

with $\theta = \frac{2\kappa}{(\kappa-1)(2+\gamma)}$. The first factor on the right is finite as it is the space used to perform the contraction. Since $H^\gamma(X)$ embeds into $L^{\frac{2}{1-\gamma}}(X)$, the second factor is also finite. A similar argument holds when $5 < \kappa \leq 9$; see e.g. [19, (4.17)]. Finally, when $3 < \kappa \leq 5$, we have $v \in L^{\frac{3}{\gamma}}([0, T]; L^{\frac{3}{2}(\kappa-1)}(X))$. Since $\frac{3}{\gamma} \geq \frac{3}{2}(\kappa-1)$ in this case, $v \in L^{\frac{3}{2}(\kappa-1)}([0, T] \times X)$ follows by applying the Hölder inequality in time.

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